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The Standard & Poor's 500 Index and The Chaotic Growth Model

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Abstract: *Standard & Poor's 500 Index (the S&P 500) includes the stocks of 500 of the most widely traded stocks in the U.S. It represents about 80% of the total value of U.S. stock markets. The basic aims of this paper are: firstly, to create the simple chaotic stock market index growth model that is capable of generating stable equilibrium, cycles, or chaos; secondly, to analyze the local stability of the S&P 500 index movements in the period 1932-1982; thirdly, to analyze the local stability of the S&P 500 index movements in the period 1982-2009; and fourthly, to discover the equilibrium levels of the S&P 500 index in the observed periods. This paper confirms the existence of the stable convergent fluctuations of the S&P 500 index in the observed periods. Further, two Elliot wave patterns were identified in the period 1932-2009. Also, the golden ratio can be used to define the equilibrium level of the S&P 500 index in the presented chaotic model.*

1. INTRODUCTION

Every stock in the S&P 500 index is shown in proportion to its total market capitalization. In other words, if the total market value of all 500 companies in the S&P 500 drops by 10%, the value of the index also drops by 10%. On the other hand, 10% movement in all stocks in the DJIA would not necessarily cause a 10% change in the index. The S&P 500 index includes companies in a variety of sectors, including energy, industrials, information technology, healthcare, financials and consumer staples. The S&P 500 uses a market capitalization weighting method, while the DJIA is a price-weighted index that gives companies with higher stock prices a higher index weighting. The S&P 500 is a member of a set of indexes created by the Standard & Poor's company. The S&P 500 (www.standardandpoors.com) is a widely quoted stock market price index, second perhaps only to the Dow Jones Industrial Average. It includes 500 large American companies traded on NYSE and Nasdaq. The S&P 500 companies are selected by an index committee that meets monthly.

Chaos theory started with Lorenz's (1963) discovery of complex dynamics arising from three nonlinear differential equations leading to turbulence in the weather system. Li and Yorke (1975) discovered that the simple logistic curve can exhibit very complex behavior. Further, May (1976) described chaos in population biology. Chaos theory has been applied in economics by Benhabib and Day (1981,1982), Day (1982, 1983), Grandmont (1985), Goodwin (1990), Medio (1993), Lorenz (1993), Jablanovic (2012, 2013, 2016), Puu, T. (2003), Zhang W.B. (2006), etc.

2. THE MODEL

The equation used to calculate the short-run aggregate supply is:

$$Y_{s,\,t} = Y_{n,t} + \alpha\,(P_t - P_t^{e}) \tag{1}$$

In the equation (1), Y is the real output, Y_n is the natural level of real output, α is the positive coefficient, P is the general price level, P_t^e is the expected general price level.

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In the short run, the supply of output depends on the natural rate of output (Y_n) and on the difference between the price level and the expected price level, (P_t - P_t^e). This relationship is expressed in the aggregate-supply equation.

The natural level of real output, Y_n , and the expected general price level , P^e , are given as:

$$Y_{n,t} = \beta Y_t \quad 0< \beta <1 \tag{2}$$

$$P_t^{\,e} = \gamma P_t \quad 0< \gamma <1 \tag{3}$$

where β and γ are the positive coefficients.

Further, it is assumed that

$$P_{s,t} = \nu P_t \quad 0< \nu <1 \tag{4}$$

where P_s is the stock price, P is the general price level.

On the other hand, GDP (Y) is the sum of consumption (C), investment (I), government purchases (G), and net export (Nx):

$$Y_{d,t} = C_t + I_t + G_t + Nx_t \tag{5}$$

In this model, the consumption function displays the quadratic relationship between consumption (C_t) and real output of the previous period (Y_{t-1}). Real output is multiplied by the coefficient μ, „the marginal propensity to consume" (MPC) . The MPC coefficient can be between zero and one.

$$C_t = \mu Y^2_{t-1} \quad 0< \mu <1 \tag{6}$$

As regards investment in period t, it is taken to be the function of change in real output in the previous period, i.e.

$$I_t = \delta Y_{t-1} \quad \delta > 0 \tag{7}$$

where δ stands for the capital –output ratio or accelerator.

Further, government spending function and net export function are given as:

$$G_t = \eta Y_{t-1} \quad 0< \eta <1 \tag{8}$$

$$N_{x,t} = \lambda Y_t \quad 0< \lambda <1 \tag{9}$$

where η and λ are the positive coefficients.

Macroeconomic equilibrium occurs when the quantity of real output demanded, Y_d, equals the quantity supplied, Y_s, or:

$$Y_{d,\ t} = Y_{s,t} \tag{10}$$

Now, putting (1), (2), (3), (4), (5), (6), (7), (8), (9) and (10) together we immediately get:

$$P_{s,t} = \left(\frac{\delta+\eta}{1-\lambda}\right) P_{s,t-1} - \left[\frac{\nu\,\mu\,(1-\beta)}{\alpha\,(\lambda-1)(1-\gamma)}\right] P_{s,t-1}{}^{2} \tag{11}$$

Further, it is assumed that the current value of the stock price is restricted by its maximal value in its time series. This premise requires a modification of the growth law. Now, the stock price growth rate depends on the current value of the stock price, P_s, relative to its maximal value in its time series $P_s^{\ m}$. We introduce p_s as $p_s = P_s / P_s^{\ m}$. Thus, p_s range between 0 and 1. Again we index p_s by t, i.e. write $p_{s,t}$ to refer to the size at time steps t = 0,1,2,3,... Now growth rate of the stock price is measured as:

$$p_{s,t} = \left(\frac{\delta+\eta}{1-\lambda}\right) p_{s,t-1} - \left[\frac{\nu\,\mu\,(1-\beta)}{\alpha\,(\lambda-1)(1-\gamma)}\right] p_{s,t-1}{}^{2} \tag{12}$$

This model given by equation (12) is called the logistic model. For most choices of α, β , γ , μ , λ, η, ν , and δ there is no explicit solution for (12). Namely, knowing α, β , γ , μ , λ, η, ν, and δ and measuring $p_{s,0}$ would not suffice to predict $p_{s,\ t}$ for any point in time, as was previously possible. This is at the heart of the presence of chaos in deterministic feedback processes. Lorenz (1963) discovered this effect - the lack of predictability in deterministic systems. Sensitive dependence on initial conditions is one of the central ingredients of what is called deterministic chaos.

3. THE LOGISTIC MAP

It is possible to show that iteration process for the logistic map

$$z_t = \pi\, z_{t-1}\,(1 - z_{t-1}),\ \pi \in [0, 4],\ z_t \in [0, 1] \tag{13}$$

is equivalent to the iteration of growth model (12) when we use the identification:

$$z_t = \left[\frac{\nu\,\mu\,(1-\beta)(1-\lambda)}{\alpha\,(\lambda-1)(1-\gamma)(\delta+\eta)}\right] p_{st} \ \text{ and } \pi = (\delta+\eta)\,/\,(1-\lambda) \tag{14}$$

Using (12) and (14) we obtain:

$$\begin{aligned} z_t &= \left[\frac{\nu\,\mu\,(1-\beta)(1-\lambda)}{\alpha\,(\lambda-1)(1-\gamma)(\delta+\eta)}\right] p_{st} \\ &= \left[\frac{\nu\,\mu\,(1-\beta)(1-\lambda)}{\alpha\,(\lambda-1)(1-\gamma)(\delta+\eta)}\right] \left\{\left(\frac{\delta+\eta}{1-\lambda}\right) p_{s,t-1} - \left[\frac{\nu\,\mu\,(1-\beta)}{\alpha\,(\lambda-1)(1-\gamma)}\right] p_{s,t-1}{}^{2}\right\} \\ &= \left[\frac{\nu\,\mu\,(1-\beta)}{\alpha\,(\lambda-1)(1-\gamma)}\right] p_{s,t-1} - \left[\frac{\nu^{2}\,\mu^{2}\,(1-\beta)^{2}\,(1-\lambda)}{\alpha^{2}\,(\lambda-1)^{2}\,(1-\gamma)^{2}\,(\delta+\eta)}\right] p_{s,t-1}{}^{2} \end{aligned}$$

On the other hand, using (13) and (14) we obtain:

$$z_t = \pi\, z_{t-1}\,(1 - z_{t-1}) =$$

$$= \left(\frac{\delta+\eta}{1-\lambda}\right)\left[\frac{\nu\,\mu\,(1-\beta)(1-\lambda)}{\alpha\,(\lambda-1)(1-\gamma)(\delta+\eta)}\right]p_{s,t-1}\left\{1-\left[\frac{\nu\,\mu\,(1-\beta)(1-\lambda)}{\alpha\,(\lambda-1)(1-\gamma)(\delta+\eta)}\right]p_{s,t-1}\right\} =$$

$$= \left[\frac{\nu\,\mu\,(1-\beta)}{\alpha\,(\lambda-1)(1-\gamma)}\right]p_{s,t-1} - \left[\frac{\nu^2\,\mu^2\,(1-\beta)^2\,(1-\lambda)}{\alpha^2\,(\lambda-1)^2\,(1-\gamma)^2\,(\delta+\eta)}\right]p_{s,t-1}^{\;2}$$

Thus we have that iterating (12) is really the same as iterating (13) using (14). It is important because the dynamic properties of the logistic equation (13) have been widely analyzed (Li and Yorke (1975), May (1976)). It is obtained that :(i) For parameter values $0 < \pi < 1$ all solutions will converge to $z = 0$; (ii) For $1 < \pi < 3{,}57$ there exist fixed points, the number of which depends on π; (iii) For $1 < \pi < 2$ all solutions monotonically increase to $z = (\pi - 1) / \pi$; (iv) For $2 < \pi < 3$ fluctuations will converge to $z = (\pi - 1) / \pi$; (v) For $3 < \pi < 4$ all solutions will continuously fluctuate; (vi) For $3{,}57 < \pi < 4$ the solution become "chaotic" which means that there exist totally aperiodic solution or periodic solutions with a very large, complicated period. This means that the path of z_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatsoever. Also, for $\pi = 2.6178$ then fluctuations will converge to $z=0.618$. The Golden Ratio (golden mean, golden number, golden proportion) is 0.618. The Fibonacci sequence, starting with zero and one, is created by adding the previous two numbers (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,144, 233, 377, …). This sequence is significant because of the golden ratio. The ratio of any number in the Fibonacci sequence relative to the number directly to its right is approximately 0.618. The Golden Ratio (golden mean, golden number, golden proportion) is 0.618. Adding the number 1 to the Golden Ratio = Phi (Φ =1.618). Both 0.618 and 1.618 are used interchangeably to represent the golden ratio because they represent the same geometric relationship. (Lidwell, Holden , and Butler, 2010, p.114).

4. EMPIRICAL EVIDENCE

The main aims of this analysis are: i) to present stock market index (the S&P 500 index) growth stability in the period 1932-1982, and ii) to present stock market index (the S&P 500 index) growth stability in the period 1982-2009. This paper uses the non-linear, logistic model (15) because stock market index is computed from the prices of the selected stocks.

In this sense,

$$s_{t+1} = \pi\, s_t - \omega\, s_t^{\,2}\, \pi,\ \omega > 0 \tag{15}$$

where s - stock market index, $\pi = (\delta+\eta) / (1-\lambda)$ and $\omega = [\nu\,\mu\,(1-\beta)] / [\alpha\,(\lambda -1)\,(1- \gamma)]$

Firstly, we transform data on stock market index from 0 to 1, according to our supposition that actual value of stock market index, S, is restricted by its highest value in the time-series, S^m. Further, we obtain time-series of $s = S / S^m$. Now, we estimate the model (15). Secondly, data on *Standard & Poor's 500 Index (the S&P 500)* are transformed (Source: http://www.macrotrends.

net/2324/sp-500-historical-chart-data) from 0 to 1, according to our supposition that actual values of *the S&P 500*, S, is restricted by its highest value in the time-series, S^m. Further, we obtain time-series of $s = S / S^m$ (see Table 1 and Table 3). Also, the Fibonacci ratios are associated to these Elliott wave patterns. It is supposed that the basic Elliott wave patterns exist in the periods: (i) Jun 1932-July 1982; and (ii) July 1982 – February 2009. The Elliot wave pattern consists of an impulse wave and a corrective wave. Impulse waves consist of 5 waves and moves in the direction of the trend. Waves 1, 3, and 5 moves in the direction of the trend, while Waves 2 and 4 move opposite to the trend. Corrective waves can be simple or complex. A simple correction consists of 3 waves (Wave A, B and C) which retrace a portion of impulse. The Elliot Wave Theory identifies (Frost A.J. & R.P. Prechter, 2006): (i) impulse waves that set up a pattern; and (ii) corrective waves that oppose the larger trend. The stock price movements are divided into: (i) trends (five waves in the direction of the main trend); and (ii) corrections (three corrective waves).

Table 1. Data on the Standard & Poor's 500 Index (the S&P 500) are transformed and Fibonacci ratios, Jun 1932-July 1982.

		the S&P 500	S/S^m	Fibonacci	ratios
	Jun-32	84.78	0.111196	0.125	1/8
1	**Apr-46**	265.38	0.348067	0.3333	1/3
2	**Jun-49**	154.2	0.202245	0.2	1/5
3	**Aug-59**	531.22	0.696802	0.7	½+1/5
4	**Oct-60**	466.31	0.611602	0.618	55/89
5	**Oct-68**	762.44	1	1	1/1
A	**Jun-70**	487.81	0.639801	0.625	5/8
B	**Dec-72**	722.94	0.948193	0.9333	3/5+1/3
C	**Jul-82**	285.93	0.37502	0.382	34/89

Source: http://www.macrotrends.net/2324/sp-500-historical-chart-data

The model (15) is estimated. The results are presented in Table 2 and Table 4.

Table 2. The estimated model (15): The S&P 500, Jun 1932-July 1982.

R=0.4982 Variance explained 24.820%		
N=8	**π**	ω
Estimate	**2.524927**	2.030342
Std. Err.	0.614166	0.729660
t(6)	4.111146	2.782585
p-level	0.006278	0.031888

From Jun 1932 to July 1982, the Standard & Poor's 500 Index (the S&P 500) moved from 84.78 to 285.93. As we can see the pattern exactly follows the wave pattern described by Elliot Wave Theory. According to the logistic equation, for $2 < \pi < 3$ fluctuations converge to $z = (\pi - 1) / \pi$, or (2.524927-1)/2.524927=0.60395. According to (14), the equilibrium value of the Standard & Poor's 500 Index (the S&P 500) was 0.60395x 2.524927 /2.030342, or 0.60395x 1.2436 or 0.7511 in the observed period. The equilibrium value of the Standard & Poor's 500 Index (the S&P 500) 0.7511 x 762.44 = 572.6687 in the observed period.

Table 3. Data on the Standard & Poor's 500 Index (the S&P 500) are transformed and Fibonacci ratios, July 1982-February 2009.

		the S&P 500	P/P^m	Fibonacci	ratios
	JuL-82	285.93	0.125099	0.125	1/8
1	**Jul-87**	728.78	0.318853	0.3333	1/3
2	**Dec-87**	557.17	0.243771	0.25	2/8
3	**May-98**	1744.22	0.763124	0.75	1/2+1/4
4	**Aug-98**	1524.95	0.66719	0.66667	2/3
5	**Aug-00**	2285.63	1	1	1/1
A	**Sep-02**	1172.37	0.512931	0.5	1/2
B	**Oct-07**	1930.53	0.844638	0.85	33/5+2/8
C	**Feb-09**	901.56	0.394622	0.4	2/5

Source: http://www.macrotrends.net/2324/sp-500-historical-chart-data

Table 4. The estimated model (15): The S&P 500, July 1982-February 2009.

R=0.58329 Variance explained 34.022%		
N=8	π	ω
Estimate	**2.628721**	2.216101
Std. Err.	0.555701	0.679140
t(6)	4.730456	3.263101
p-level	0.003223	0.017182

From July 1982 to February 2009, the Standard & Poor's 500 Index (the S&P 500) moved from 285,93 to 901.56. As we can see the pattern exactly follows the wave pattern described by Elliot Wave Theory. According to the logistic equation, for $2 < \pi < 3$ fluctuations converge to $z = (\pi - 1) / \pi$, or (2.628721-1)/2.628721=0.61959. According to (14), the equilibrium value of the Standard & Poor's 500 Index (the S&P 500) was 0.61959 x 2.628721/2.216101, or 0.61959x 1.18692 or 0.7354 in the observed period. The equilibrium value of the Standard & Poor's 500 Index (the S&P 500) was 0.7354 x 2285.63= 1680.8523 in the observed period.

5. CONCLUSION

This paper suggests conclusion for the use of the chaotic stock price growth model in predicting the fluctuations of the Standard & Poor's 500 Index (the S&P 500). The model (12) has to rely on specified parameters α, β, γ, μ , λ, η, ν , and δ , and initial value of the stock price, and /or the Standard & Poor's 500 Index (the S&P 500), $p_{s\ 0}$. But even slight deviations from the values of parameters: α, β, γ , μ , λ, η, ν , and δ , and initial value of the stock price and/or the Standard & Poor's 500 Index (the S&P 500), $p_{s\ 0}$, show the difficulty of predicting a long-term stock price and / or Standard & Poor's 500 Index (the S&P 500).

A key hypothesis of this work is based on the idea that the coefficient $\pi = (\delta+\eta) / (1-\lambda)$ plays a crucial role in explaining local stability of the stock price, and/or the Standard & Poor's 500 Index (the S&P 500), where, δ is the capital –output ratio or accelerator η is the government spending ratio, λ is the net export ratio. The estimated value of the coefficient π was **2.524927** in the period Jun 1932 - July 1982. The equilibrium price of the Standard & Poor's 500 Index (the

S&P 500) was 572.6687 in the observed period Also, the Elliott wave pattern was observed. The Fibonacci ratios are included in the model. The equilibrium value in the modified model was 0.60395, while the golden ratio is 0. 618. The estimated value of the coefficient π is **2.628721** in the period July 1982 - February 2009. The equilibrium price of the Standard & Poor's 500 Index (the S&P 500) was 1680.8523 in the observed period. Also, the second Elliott wave pattern is observed. The Fibonacci ratios are included in the model. The equilibrium value in the modified model was 0.61959, while the golden ratio is 0.618.

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